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Coordination of a Multidivisional Organization Through Two Levels of Management

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This paper develops the bilevel multidivisional programming problem (BMPP) as a model for a decentralized organization. In particular, a hierarchical arrangement comprising one superior unit and *M* subordinate units is proposed where each unit is assumed to control a unique set of decision variables defined over jointly dependent strategy sets. Individual but interdependent objective functions are also assumed so that each unit may influence but not control another. In the formulation, decisions are made in two stages with top management given the first choice followed by the concurrent responses of the divisions. In this way it is possible to account for production externalities at the lower level and coordination activities at the higher level. As such, after outlining the major geometry of the linear case, the usefulness of the general model is discussed from a management perspective. Ramifications relating to opportunity costs, capacity utilization and computational requirements are presented and highlighted through a variety of examples.

1. INTRODUCTION

DECISION-MAKING in large, hierarchical organizations rarely proceeds from a single point of view. Two of the most prominent aspects of such organizations are specialization closely followed by coordination [13]. The former arises from a practical need to isolate individual jobs or operations and to assign them to specialized units. This leads to departmentalization, however, to accomplish the overall task, the specialized units must be coordinated. The related process, also referred to in organization theory as control, divides itself quite naturally into two parts: the establishment of individual goals and operating rules for each member and the enforcement of these rules within the work environment. The first is referred to as control-in-the-large and deals with the selection of appropriate divisional or lower level performance criteria and, more generally, the selection of the modes of coordination. The second is called control-in-the-small and relates to the choice of coordination inputs.

An important control variable in the theory of departmentalization is the degree of self-containment of the organization units [9, 13]. A unit is self-contained to the extent and degree that the conditions for carrying out its activities are independent of what is done elsewhere in the system. The corporate or higher level unit is then faced with the coordination problem of favorably resolving the divisional unit interactions. Mathematical programming has often been used as the basis for modeling these interactions with decomposition techniques (see, e.g., [6, 9, 12]) providing solutions for problems of large scale. The central idea underlying these techniques is very simple and can be envisioned as the following algorithmic process: top management, with its set of goals, asks each division of the company to calculate and submit an optimal production plan as though it were operating in isolation. Once the plans are submitted, they are modified with the overall benefit of the company in mind. Marginal profit figures are used to successively reformulate the divisional plans at each stage in the algorithm.

An output plan ultimately emerges which is optimal for the company as a whole and which therefore represents the solution to the original programming problem.

Although this procedure attempts to mimic corporate behavior it fails on two counts. The first relates to the assumption that it is possible to derive a single objective or utility function which adequately captures the goals of both top management and each subordinate division. The second stems from lack of communications among the components of the organization; at an intermediary stage of the calculations there is no guarantee that each division's plans will satisfy the corporate constraints. In particular, if the production of some output by division k imposes burdens on other divisions by using up a scarce company resource, or by causing an upward shift in the cost functions pertaining to some other company operation, division k's calculation is likely to lead it to overproduce this item from the point of view of the company because the costs to other divisions will not enter its accounts. This is the classical problem of external diseconomies. Similarly, if one of division k's outputs yields external economies where a rise in its production increases the profitability of other divisions, division k may (considering just its own gains in its calculations) not produce enough of this product to maximize the profits of the firm. This may result in a final solution that does not realistically reflect the production plan that probably would have been achieved had each division been given the degree of autonomy it exercises in practice.

Another way of treating the multi-level nature of the resource allocation problem is through goal programming. Ruefli [16] was the first to apply this technique by proposing a generalized goal decomposition model. Freeland & Baker [11] expanded on this work and developed a model capable of representing a wide range of operational characteristics including informational autonomy, interdependent strategies, and 'bounded rationality' [13] or individual goals. Others [7, 9, 14] have used combinations of the above strategies to solve problems related to government regulation, distribution and control. The approach that we follow derives from the complementary notions of two-stage optimization [1, 2, 3, 4, 5] and equilibrium analysis [20]. That is, decision-making between levels is assumed to proceed sequentially but with some amount of independence to account for the divergence of corporate and subordinate objectives. At the divisional level each unit simultaneously attempts to maximize its own production function and, in so doing, produces a balance of opposing forces.

In the next section the structure of the bilevel multidivisional programming problem (BMPP) is examined and a precise mathematical definition given. In Section 3 the relationship between the multidivisional and equilibrium problem is discussed. A brief evaluation of the applicability of other multicriteria decision-making techniques is also offered. Section 4 unifies the theories of equilibrium and bilevel programming while describing the management implications of the subsequent formulation, and outlining a number of solution techniques. Basic results are presented that, under certain conditions, establish the equivalence of the BMPP with a standard mathematical program. A linear case is examined in detail. Section 5 continues with a discussion of capacity utilization and contrasts the centralized and decentralized approaches by way of example. Finally, some of the geometric properties of the linear BMPP are derived in the Appendix.

2. DEVELOPMENT OF THE MODEL

A distinguishing characteristic of multi-level systems is that the decision maker at one level may be able to influence the behavior of a decision maker at another level but not completely control his actions. In addition, the objective functions of each unit may, in part, be determined by variables controlled by other units operating at parallel or subordinate levels. For example, policies effected by corporate management relating to resource allocation and benefits may curtail the set of strategies available to divisional management. In turn, policies adopted at the lower levels affecting productivity and marketing may play a role in determining overall profitability and growth. Bialas & Karwan [4] have noted the following common features of multi-level organizations:

(a) interactive decision-making units exist within a predominantly hierarchical structure;

(b) each subordinate level executes its policies after, and in view of, decisions made at a superordinate level;

(c) each unit independently maximizes net benefits, but is affected by the actions of other units through externalities;

(d) these extramural effects enter a decision maker's problem through his objective function and feasible strategy set.

The need for specialization and decentralization has traditionally been met by the establishment of profit centers. In this context, divisions or departments are viewed as more or less independent units charged with the responsibility of operating in the best possible manner so as to maximize profit under the given constraints imposed by the corporate management. The problem of decentralization is essentially how to design and impose constraints on the department units so that the well-being of the overall corporation is assured. The traditional way to coordinate decentralized organizations is by means of the pricing mechanism; coordination is designed by analogy with the operation of a free market or competitive economy. Exchange of products between departments is allowed and internal prices are specified for the exchange commodities. The problem of effective decentralization reduces then to the selection of the internal prices.

The above features are incorporated in the model we propose. The framework embodies a corporate management unit at the higher level and M divisions or subordinate units at the lower level. The latter may be viewed as either separate operating divisions of an organization or coequal departments within a firm, such as production, finance and sales. This structure can be extended beyond two levels (e.g., see [7, 8, 9]) with the realization that attending behavioral and operational relationships become much more difficult to conceptualize and describe.

To formulate the problem mathematically, suppose the higher level decision maker wishes to maximize his objective function F and each of the M divisions wishes to maximize its own objective function f^i. Control of the decision variables is partitioned among the units such that the higher level decision maker may select a vector $x^0 \in S^0 \subset R^{n^0}$ and each lower level decision maker may select a vector $x^i \in S^i \subset R^{n^i}$, $i = 1, \ldots, M$. Calling $x \equiv (x^0, x^1, \ldots, x^M)$ and letting

$$n = \sum_{i=0}^{M} n^i,$$

in the most general case we have $F, f^1, \ldots, f^M: R^n \to R^1$. It shall be assumed that the corporate unit has the first choice and selects a strategy $x^0 \in S^0$, followed by the M subordinate units who select their strategies $x^i \in S^i$, simultaneously. In addition, the choice made at the higher level may affect the set of feasible strategies available at the lower level, while each lower level decision maker may influence the choices available to his peers. The strategy sets will be given the following explicit representation:

$$S^0 = \{x^0 : g^0(x^0) \leqq 0\}$$

$$S^i = \{x^i : g^i(x) \leqq 0\}, \quad i = 1, \ldots, M$$

where $g^0: R^{n^0} \to R^{m^0}$ and $g^i: R^n \to R^{m^i}$, $i = 1, \ldots, M$. Notice that g^i is a function not only of x^i but each of the other decision variables, call them $x^{\bar{i}}$. This suggests the useful notation $x \equiv (x^i, x^{\bar{i}})$.

In order to ensure that the problem is well posed it shall be assumed through the remainder of this paper that all the functions are twice continuously differentiable and that the sets S^i, $i = 1, \ldots, M$, are nonempty and compact; i.e. the ith unit always has some recourse. With these assumptions the BMPP can now be defined:

$$\max_{x^0} (F(x^0, x^{\bar{0}}) : g^0(x^0) \leqq 0) \qquad (1.1)$$

where

$$x^{\bar{0}} \equiv (x^1, \ldots, x^M) \text{ solves}$$

$$\left.\begin{array}{l} \max_{x^i} f^i(x^i, x^{\bar{i}}) \\ \text{subject to } g^i(x^i, x^{\bar{i}}) \leqq 0 \end{array}\right\} i = 1, \ldots, M. \qquad (1.2)$$

If $M = 0$, problem (1) reduces to a standard mathematical program; if $M = 1$ the bilevel programming problem (BLPP) results; if (1.1) is removed we are left with an equilibrium programming (EP) problem [20]. The latter will be discussed in the next section. The meaning of the expression 'where $x^{\bar{0}}$ solves' derives from noncooperative game theory and has come to be used in the definition of a solution to the EP. This will be taken up in Section 3; the definition of a solution to the BMPP will be given in Section 4.

3. THE MULTIDIVISIONAL PROBLEM AND EQUILIBRIUM PROGRAMMING

For the moment let us assume that a decision has been made by the corporate unit so x^0 is now fixed for problem (1). In light of this decision each of the M divisions must concurrently select their strategies so that each objective function in (1.2) is maximized. But what does this mean? The common notion of an optimum is based on a single objective, whereas in our problem a favorable result for one division may be highly unfavorable for another. We are therefore forced to consider a different concept that leads to the idea of equilibrium. As has already been noted, in reality parallel divisions are often in silent competition. In a decentralized system each unit is allowed to make some decisions but there is no single decision maker who can dictate the final course of action. Although power may not be equally distributed among the divisions, no one unit has complete control.

In order to define a solution to (1.2) for x^0 fixed we introduce the idea of equilibrium which is commonplace in such fields as economics and the physical sciences. Essentially, at equilibrium each decision maker attains the best result possible. A balance of opposing forces is produced so there is no desire on the part of any individual unit to alter its choice (we are tacitly excluding the formation of coalitions and the use of side payments to improve individual payoffs). After rewriting each component of (1.2) as follows:

$$\max_{x^i} (f^i(x^i, x^{\bar{i}}) : g^i(x^i, x^{\bar{i}}) \leqq 0) \qquad (2)$$

and calling it subproblem i, we may state:

Definition 1. Given $\bar{x}^{\bar{i}}$, if $\bar{x}^i$ solves subproblem i $(i = 1, \ldots M)$, then $\bar{x} = (\bar{x}^i, \bar{x}^{\bar{i}})$ is called an *equilibrium point*.

Explicitly then, an equilibrium point $\bar{x} = (\bar{x}^i, \bar{x}^{\bar{i}})$ solves (2) for $i = 1, \ldots, M$ so we are at a point of stability. No incentive exists for any of the divisions to deviate from $\bar{x}^i$ because each has optimized its individual objective function. To avoid difficulties relating to the existence of a solution it shall be assumed that for $\bar{x}^{\bar{i}}$ fixed at its optimum, $\bar{x}^i$ uniquely solves the problem faced by division i. If this were not true there might exist additional points x^i that solve (2) but do not necessarily yield unique objective function values for the other subproblems even when $\bar{x}^i$ remains unchanged (see [3] for a further discussion of this matter).

An alternative formulation of the BMPP could have been achieved by using the techniques common to multicriteria decision-making [15,18]. Two such methodologies that have been applied successfully follow completely divergent paths. In one extreme, the problem is viewed mostly from the perspective of a central decision maker whose subjective value judgements form the basis of all major action. The task then is to translate these judgements into some form of preference function. While this approach may be most suitable for use in less complex problems where decision criteria can, themselves, be treated as independent factors, its power is lost in the conflictual setting. The high level of aggregation required in the formation stage effectively precludes the ability to confront the internal structure of the system directly. Perhaps the weakest feature of the preference approach, though, is the necessity of developing a hypothetical utility function that will accurately measure the concerns of each parallel unit.

At the other extreme, the problem is viewed objectively by the systems analyst who then gives a multiobjective formulation which attempts to capture the relevant characteristics of the entire organization. The resultant model is then optimized in the sense that a set or subset of efficient solutions is found. The term 'vector optimization' appropriately reflects the nature of this approach: the corresponding problem can be stated as follows [19]: given a vector-valued criteria function $f(x) = [f^1(x), \ldots, f^M(x)]$ defined on the set X where $X = \{x \in R^n : g_j(x) \leqq 0, j = 1, \ldots, m\}$ find all efficient points.

Definition 2. A *point* x^0 is said to be *efficient* if $x^0 \in X$ and there exists no other feasible point $x \in X$ such that $f(x) \geqq f(x^0)$ and $f(x) \neq f(x^0)$.

Succinctly, the problem may be written as:

$$V \max: f(x) \text{ subject to } x \in X$$

where the multidimensional aspects of the decision criteria are now explicitly taken into account. The difficulty arising from this formulation, however, is that one person is still called upon to make the final selection. The set of efficient solutions rarely contains only one element so the decision maker is faced with the impossible task of rendering an impartial decision. Even if a lexicographical ordering could be established among the set of objectives represented by f, the uncooperative nature of the real system suggests, in fact, that the true solution may not even be efficient (e.g., see [19]).

The persuasiveness of these arguments, coupled with recent advances in bilevel programming, have led us to the BMPP formulation. In the next section, we establish some of the properties of this problem and describe a number of solution methodologies.

4. MANAGERIAL AND COMPUTATIONAL IMPLICATIONS

In order to appreciate the management implications of the BMPP, it is first necessary to transpose it into a more familiar form. The approach that we follow is based on the Kuhn–Tucker system of equations associated with the M subproblems given in (2). If problem (1) is tentatively viewed from the position of the higher level decision maker, the constraint region (1.2) can be thought of as being implicitly defined by a string of optimization problems. This interpretation is still difficult to work with, however, but it does suggest an alternative representation of (1.2) admitting a standard mathematical programming formulation of (1). To some extent this can be realized by appending the lower level decision maker's Kuhn–Tucker conditions to the higher level decision maker's feasible set. In the presence of certain regularity conditions it can be shown that the solution to the resultant problem would also be a solution to the BMPP. Theorem 1 below summarizes this situation. The proof will be omitted because it closely parallels that given in [3]. It should be noted that Aiyoshi & Schimizu [1] also investigated a formulation similar to (1) but failed to recognize the transformation given by (3) below. Before stating the theorem, however, let us define what is meant by a solution to the BMPP:

Definition 3. Let $x^* = (x^{*0}, x^{*\bar{0}}) \in S$

where

$$S = \bigcap_{i=0}^{M} S^i.$$

Now x^* will be called a solution to the BMPP if:

(a) $x^{*\bar{0}}$ is an equilibrium point of (1.2) for x^{*0} fixed;

(b) for all $\bar{x} \in S$ such that (a) is true, $F(x^*) \geqq F(\bar{x})$.

Theorem 1

Let a constraint qualification hold for each subproblem i in (1.2) when $x^{\bar{i}}$ is fixed. A necessary condition that x^* solves (1) is that there exists a $w^* \in R^m$ such that (x^*, w^*) solves:

$$\max_{x,w} F(x) \tag{3.1}$$

subject to:

$$g^0(x^0) \leqq 0 \tag{3.2}$$

$$\left.\begin{aligned} \nabla_{x^i} f^i(x) - w^i \nabla_{x^i} g^i(x) &= 0 & (3.3)\\ w^i g^i(x) &= 0 & \\ g^i(x) &\leqq 0 & (3.4)\\ g^i(x) &\leqq 0 & (3.5)\\ w^i &\geqq 0 & (3.6) \end{aligned}\right\} i = 1, \ldots, M$$

where w^i is the m^i-dimensional vector of dual variables (shadow prices) associated with subproblem i, $w \equiv (w^1, \ldots, w^M)$, and ∇ is the gradient operator.

In general the solution to (3) provides an upper bound on the corporate objective function F. If further restrictions are placed on the problem functions, however, sufficient conditions for optimality can be stated. We assume here that (1) has a solution, but as mentioned above this might not always be true.

Corollary 1

Let the conditions of Theorem 1 hold and assume that f^i and $-g^i$ are concave in x^i for $x^{\bar{i}}$ fixed, $i = 1, \ldots, M$. A necessary and sufficient condition for x^* to solve (1) is that there exists a w^* such that (x^*, w^*) solves (3).

An important implication of Corollary 1 is that it gives us the opportunity to obtain the shadow prices relating not only to the corporate objective but to each division's objective as well. A solution to (3) implicitly provides the standard set of dual variables, v^i $(i = 1, \ldots, M)$, accompanying constraint (3.5), while concurrently yielding sensitivity information about division i's payoff with respect to incremental changes in its resources by explicitly computing w^i. In mathematical terms, if we let $g^i(x) = \bar{g}^i(x) - b^i$, then $v^i = \partial F / \partial b^i$ componentwise while $w^i = \partial f^i / \partial b^i$. As a consequence, resource allocation decisions can be made in full view of their expected effects on both the corporate and divisional objectives.

In general, the relationship between w^i and v^i is not easy to establish. In situations where degeneracy exists (a problem common to the decompositional formulation [12]) it is quite probable that one or more of the components of w^i will be greater than zero while the corresponding components of v^i will be equal to zero. From the corporate point of view, such allocations may not be desirable since they serve only to further a particular division's objective without improving the overall position of the firm.

In Corollary 1, explicit in the statement that x^* solves (1) is the fact that $x^{*\bar{0}}$ is an equilibrium point of the M subproblems given in (2). In [20], Zangwill & Garcia specifically address the problem of finding such a point after similarly deriving the above Kuhn–Tucker system of necessary conditions. They contend that path following, a general technique for solving sets of nonlinear equations, can be used with great advantage to obtain equilibrium points. While this may be true for the simple EP, the presence of a higher level decision maker in our problem severely undermines its application here. Problem (3) will therefore be treated as a nonlinear program. The fact that it is inherently nonconvex does foreshadow difficulties, but, for some special cases (see [3, 4, 5]) solutions will be obtained directly. In particular, when all the functions in (1) are affine we get the linear BMPP:

$$\max_{x^0} (C^0x : A^0x^0 \leqq b^0) \tag{4.1}$$

where $x^{\bar{0}}$ solves:

$$\left.\begin{array}{l} \max\limits_{x^i} C^ix \\ \text{subject to } A^ix \leqq b^i \end{array}\right\} i = 1, \ldots, M \tag{4.2}$$

where $C^i \in R^n$, $b^i \in R^{M^i}$, $i = 0, \ldots, M$; $A^0 \in R^{m^0 x n^0}$, $A^i \in R^{m^i x n}$, $i = 1, \ldots, M$ are constants and any nonnegativity constraints are subsumed in the appropriate matrices.

A solution to (4) can be obtained by recasting it in the form of (3) and invoking Corollary 1. If this is done, the resultant program would be linear in x and w save the complementarity constraint (3.4); however, it is shown in the Appendix that the optimum of (4) occurs at a vertex of the explicit constraint region $S = \{x : A^ix \leqq b^i, i = 0, 1, \ldots, M\}$. For purposes of planning, this result is noteworthy because it indicates the number of activities that will be binding at the solution. Such information is often useful in determining capacity utilization and in making capital budgeting decisions. From a strictly computational point of view, its significance becomes apparent when considering the possible application of linear programming techniques. In fact, a number of algorithms [4, 5] have already been developed for the BLPP to take advantage of this special structure and could well be adapted to the linear BMPP. Those techniques which have been successfully exploited include implicit basis search, vertex enumeration, and complementary pivoting, with the latter proving to be the most efficient.

The following example illustrates the basic properties of the linear BMPP. It should be noted that, in general, the solution to (1) regardless of functional form will not be Pareto-optimal (i.e. efficient). This results from the fact that cooperation between and among levels has been expressly excluded from the model.

Example

Letting

$$x^0 = (x_{01}, x_{02}), \quad x^1 = (x_{11}, x_{12}), \quad x^2 = x_2$$

and

$$w^1 = (w_{11}, w_{12}, w_{13}, w_{14}), \quad w^2 = (w_{21}, w_{22}, w_{23}):$$

$$\max_{x^0 \geqq 0} (-2x_{01} + x_{02} + 3x_{11} - x_2 : x_{01} + 2x_{02} \leqq 4)$$

where x^1 solves

$$\max_{x^1 \geqq 0} (x_{01} + 2x_{11} + 3x_{12} + x_2),$$

subject to

$$x_{01} + 2x_{11} - x_{12} \leqq 6$$
$$x_{12} + x_2 \leqq 9;$$

and x^2 solves

$$\max_{x^2 \geqq 0} (-x_{02} + 2x_2)$$

subject to

$$-2x_{02} + x_{11} - 3x_2 \leqq 12$$
$$-x_{12} + 4x_2 \leqq 8.$$

Solution:

$$\hat{x}^0 = (0, 2), \; \hat{x}^1 = (5.8, 5.6), \; \hat{x}^2 = (3.4);$$
$$\hat{w}^1 = (1, 4, 0, 0), \; \hat{w}^2 = (0, 0.5, 0);$$
$$\hat{v}^1 = (2, 1.5, 0, 0), \; \hat{v}^2 = (0, 0, 0);$$
$$\hat{F} = 16, \hat{f}^1 = 28, \hat{f}^2 = 4.8.$$

The solution was obtained by first transposing the problem to resemble (5) but with (5.4) reformulated as a piecewise linear, separable function, and then using a general branch and bound algorithm that finds global optima to nonlinear, separable programs (see [3]). A check of the above feasible regions at the solution indicates that five binding constraints exist and that their gradients are linearly independent. Because S, the joint constraint region, is a polyhedron in R^5, it follows that the solution is a vertex of S. This verifies Theorem 3 of the Appendix. Further, an examination of the complementary sets of shadow prices brings to light the differences in marginal expectations. For Division 1, a one unit increase in the first resource will produce one additional unit of profit at the divisional level and two additional units of profit at the corporate level. For the second resource the ratio is reversed; one additional unit yields a divisional improvement of four units but a corporate improvement of only 1.5 units. At Division 2, the shadow prices for the first resource are both zero while those for the second are 0.5 and 0, respectively. This means that a unit change here will produce a 0.5 unit change in payoff to the division but no change at the corporate level. The implication of this result is that it may be possible to reduce the availability of the second resource to Division 2 without adversely affecting the objective of the firm as a whole.

Finally, it can be seen that if management attempts to maximize its overall objective function, F, subject to the divisional constraints, but without regard to their specific objectives, the following settings result:

$$\underline{x}^0 = (0, 2), \quad \underline{x}^1 = (7.5, 9), \quad \underline{x}^2 = (0);$$
$$\underline{F} = 24.5, \quad \underline{f}^1 = 42, \quad \underline{f}^2 = -2.$$

This solution proves to be quite satisfactory to the corporate unit as well as Division 1 because both their payoffs improve substantially; however, it fails to place the system in equilibrium. Division 2, in an effort to improve $\underline{f}^2$, will now exploit the slack in its second constraint by increasing the setting of x_2 from its current value of 0, up to 4.25. This will subsequently induce a violation in Division 1's second constraint thereby initiating a reduction, first in x_{12}, next in x_{11}, and ultimately in x_2. This pattern may continue indefinitely, or terminate should an equilibrium point exist. For this particular example, the cycle eventually converges to the exact solution because the centralized and decentralized settings, denoted by $\underline{x}^0$ and $\hat{x}^0$ above, coincide. Note that the stipulation of non-cooperation, partially realized through the limited autonomy granted the divisions, prevents the achievement of the centralized solution despite the fact that its collective payoffs exceed those obtained from the decentralized formulation. Nevertheless, without the introduction of side payments or correlated strategies, top management's ability to directly control outcomes at either level in the organization is measurably weakened by the assumed reward system and hierarchical structure in place.

Finally, it should be mentioned that whatever disadvantages exist in using the algorithm referenced in [3] for solving the linear BMPP derive from the general inefficiencies associated with all branch and bound techniques. Although it may be possible to accelerate its convergence by exploiting the linear structure of (5), this remains to be done. A second approach which offers a similar promise involves the transformation of the linear BMPP into a bilinear programming problem by moving the complementarity term (5.4) from the constraint region to the objective function (5.1) giving:

$$\max_{x,w} \left(C^0 x - K\left(\sum_{i=1}^{m} w^i s^i \right) \right) \tag{9}$$

subject to (5.2), (5.3), (5.5) and (5.6)

where K is a large positive constant and $s^i = A^i x - b^i$ is an m^i-dimensional vector of surplus variables, $i = 1, \ldots, M$. Although problem (9) is still inherently nonconvex, it may be possible to solve it efficiently by adapting an extant algorithm (e.g [17]) to take advantage of the special nature of the bilinear term.

5. EFFECTIVE CAPACITY UTILIZATION

High productivity in any system requires that all resources be utilized in the most effective manner. In a linear programming framework this is not always assured because the designing of an optimal production mix (rather than finding a solution to the more common problem where the resource vector is taken as fixed) frequently leads to severe degeneracy. A second drawback of the traditional LP approach concerns the presence of multiple optimal solutions, only one of which may provide the best use of company resources. In the following example, we demonstrate the nature of this problem and show how it may be overcome with the BMPP formulation. The Birch Paper case [10], which has often been used to illustrate other problems of decentralization, will serve as the model.

The Birch Paper Company is an integrated firm producing paperboard at its Southern Division, corrugated boxes at its Thompson Division and specialized paper products at its Northern Division. Each unit is judged independently on the basis of its profit and return on investment despite the fact that much of its business is intra-company. Specifically, Northern markets special display boxes which it designed in conjunction with the Thompson division, who spent months perfecting manufacturing procedures and tooling up for production. Thompson, which sells boxes both to Northern and to outside firms may buy its paperboard from either an outside supplier or Southern, who in turn is free to market either to Thompson or to other manufacturers.

The corporate unit has little control over divisional schedules, but may establish the internal transfer prices (which are assumed to affect production capacity as well as divisional profits) in an attempt to influence the final product mix. Southern's out-of-pocket costs are $168 (all costs and prices are quoted for 1000 boxes) and faces a market price of $280. In addition, it may sell paperboard to Thompson for P_1. The latter incurs $120 in costs processing the paperboard into boxes which it can sell for $450. Alternatively, Thompson may go to an outside supplier for $270. Northern values its special display boxes at $480 and may purchase its basic materials from either Thompson at a price P_3 or from an outside firm, The Eire Company, for $430. If this route is followed, Thompson will buy 'outside' liners from Southern for P_2 and then sell them to Eire for $120 after incurring an additional $25 for coloring and printing.

This entire arrangement is shown in Fig. 1. In all, 11 variables have been identified at the divisional level and are listed in Table 1 along with their contribution margins and attendant costs. This information, coupled with a number of restrictions on transfer pricing, leads to the following BMPP.

Corporate unit

Any number of objectives could be proposed for the firm as a whole, ranging from maximizing the difference between revenues and costs to maximizing the flow of goods through the organization or maximizing the output of a particular division. For purposes of illustration, the latter has been adopted for Northern. In so

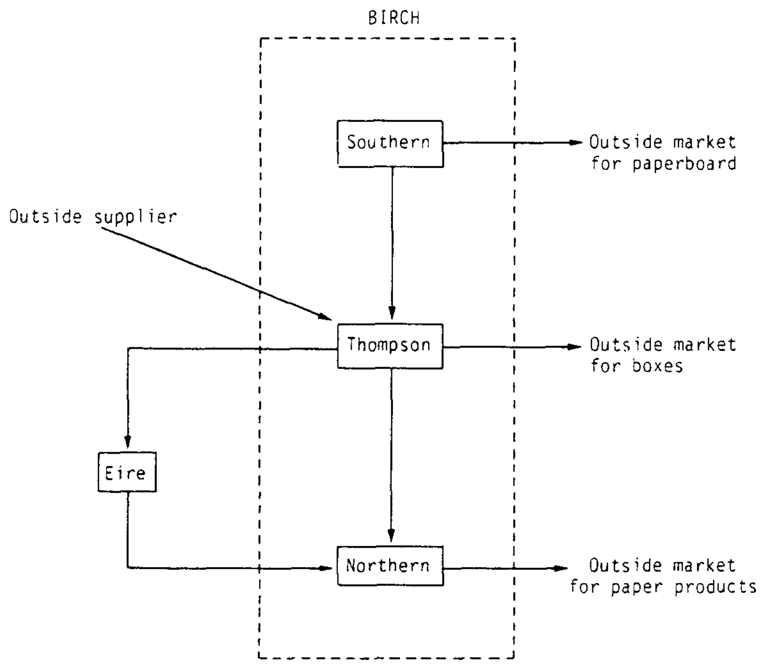

FIG. 1. *The Birch Paper Company.*

TABLE 1. PRODUCTION OPTIONS AND COST DATA

Division	Product	Purchased from	Sold to	Variable name	Outside selling price[1]	Transfer price	Cost to division[1]	Contribution margin to the division
Southern	Paperboard	—	Thompson	S_1	—	P_1	168	$P - 168$
	Liners	—	Thompson	S_2	—	P_2	54	$P_2 - 54$
	Paperboard	—	Outside	S_3	280	—	168	112
Thompson	Boxes	Southern	Northern	T_1	—	—	$P_1 + 120$	$P_3 - P_1 - 120$
	Liners	Southern	Eire	T_2	120	—	$P_2 + 25$	$95 - P_2$
	Boxes	Southern	Outside	T_3	450	—	$P_1 + 120$	$330 - P_1$
	Boxes	Outside	Northern	T_4	—	P_3	390	$P_3 - 390$
	Boxes	Outside	Outside	T_5	450	—	390	60
Northern	Paper products	Thompson	Outside	N_1	480	—	P_3	$480 - P_3$
	Paper products	Eire	Outside	N_2	480	—	432	48

[1] All prices and costs are per thousand boxes.

doing, we will impose the restriction that the transfer prices, $P = (P_1, P_2, P_3)$, must be set so that the contribution margins shown in the last column of Table 1 for each division are non-negative. This leads to the following corporate model:

$$\max_{P} N_1 + N_2$$

subject to:

$$P_1 - P_3 \leqq -120,$$
$$168 \leqq P_1 \leqq 330,$$
$$54 \leqq P_2 \leqq 95,$$
$$390 \leqq P_3 \leqq 480;$$

where:

P_1 = the price Southern charges Thompson for paperboard,

P_2 = the price Southern charges Thompson for 'outside' liners,

P_3 = the price Thompson charges Northern for boxes.

Notice that the objective function above does not depend directly on the choices of P but only on Northern's decision variables. The implicit nature of this relationship will become more evident upon examining the divisional problems.

Southern Division

All three divisions are considered profit centers with objective functions formulated accordingly. Individual capacities are restricted by the amount of activity required to produce the respective products. For Southern, it is assumed that its capacity expands directly with the transfer price P_1 to reflect an increasing supply curve. The following model results:

$$\max_{S \geqq 0} (P_1 - 168)S_1 + (P_2 - 54)S_2 + 112\, S_3$$

subject to:

$$S_1 + \tfrac{1}{3}S_2 + S_3 \leqq P_1/1.68,$$
$$S_1 - T_1 - T_3 = 0,$$
$$S_2 - T_2 = 0.$$

The first constraint is a production limitation with a minimum right hand side value of 100 (realized when P_1 is set equal to its lower bound). The second two constraints derive from material balance considerations.

Thompson Division

Thompson's problem is similar to Southern's but with the additional restriction that it faces a maximum exogenous demand of 50,000 boxes. This leads to:

$$\max_{T \geqq 0} (P_3 - P_1 - 120)T_1 + (95 - P_2)T_2 + (330 - P_1)T_3 + (P_3 - 390)T_4 + 60T_5$$

subject to:

$$T_1 + \frac{1}{7}T_2 + T_3 + T_4 + T_5 \leqq P_3/3.9,$$
$$T_3 + T_5 \leqq 50,$$
$$-S_1 + T_1 + T_3 = 0,$$
$$-S_2 + T_2 = 0.$$

Once again, the first two constraints are capacity and external sales restrictions, respectively, and the last two are the material balance equations.

Northern Division

Northern is assumed to have a fixed capacity of 150 and will choose its product mix primarily on the basis of the price, P_3, indirectly forwarded by Thompson. The following problem results:

$$\max_{N \geqq 0} (480 - P_3)N_1 + 48\, N_2$$

subject to:

$$N_1 + N_2 \leqq 150,$$
$$-T_1 - T_4 + N_1 = 0,$$
$$-T_2 + N_2 = 0,$$

where the first constraint represents the capacity limitation and the second two repeat the material balance requirements.

It is instructive to examine both the centralized and decentralized versions of this problem. A solution to the former is obtained by simply maximizing $N_1 + N_2$ over all the constraints. However, because multiple optima are present there is no guarantee that the solution realized will satisfy secondary corporate goals such as full capacity utilization. A typical result might be:

$$P = (350, 54, 450), \ S = (109.5, 40.4, 0),$$
$$T = (109.6, 40.4, 0, 0, 0), \ N = (109.6, 40.4);$$
$$F = 150, \quad f^1 = 6795, \quad f^2 = 1656, \quad f^3 \quad 5208,$$

which is seen to evidence 73.3 units of slack at Southern and potentially another 7.7 units at Thompson should P_3 be increased to \$480.

By way of contrast, the BMPP formulation takes into consideration the goals of each of the divisions and therefore yields a roundly better result:

$$P = (330, 54, 480), \ S = (21.7, 128.3, 117.2),$$
$$T = (21.7, 128.3, 0, 0, 50), \ N = (21.7, 128.3);$$
$$F = 150, \quad f^1 = 14474, \quad f^2 = 8911, \quad f^3 = 6158.$$

From these settings we see that the system is now producing at maximum capacity and is in complete equilibrium. Finally, notice that if the slack were removed from the first solution by increasing S_3 to 73.3, P_3 to \$450 and hence T_5 to 7.7, the decentralized solution would still be uniformly more efficient.

6. CONCLUSIONS

The basic problem of decentralized control, as viewed from the corporate level, is one of inducing its subordinate units to increase those activities which yield external economies and to decrease those which produce external diseconomies by just the right amount. While the standard techniques of decomposition have been used successfully in many areas, they generally fail to address this problem in full. The model we have developed makes up for many of their inherent shortcomings by recognizing that the decision process in a hierarchical decentralized organization is both sequential and multiobjective. In most such organizations, it is either impractical or impossible for the higher level decision maker to impose his utility function on the subordinate divisions. The main feature of the proposed model is that it provides pairwise sensitivity information between corporate and lower level payoffs, while permitting each unit to pursue its individual set of goals.

The bilevel multidivisional programming problem, which grew out of the modeling process, was fashioned from a combination of bilevel and equilibrium programming considerations. The final program, however, is highly nonlinear and inherently nonconvex so a great deal of work may be required to find its solution. Nevertheless, a number of special cases readily lend themselves to proven computational techniques, even in large scale. The linear BMPP, in particular, exhibits a series of properties common to the standard linear program which may be easily exploited in algorithmic development.

The more general case is likely to prove more difficult to handle. It may be possible to devise an iterative scheme where information is passed back and forth between the two levels producing adjustments in strategies which eventually lead to a solution. Path following methods might be effectively employed for the computations associated with the lower level. Nevertheless, the best hope for solving the general BMPP probably rests with recent advances in bilevel programming. At a minimum, if it is not possible to obtain a global solution, it should at least be possible to derive upper and lower bounds on unit objective functions and then perhaps use a goal programming scheme to improve the results. A number of such approaches are now being investigated.

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APPENDIX

THE GEOMETRY OF THE LINEAR BMPP

The first property that we develop relates to the constraint region of the higher level decision maker. By noting that each subproblem in (4.2) is a linear program with corresponding feasible set $S^i = \{x^i : A^i x \leqq b^i\}$, it can be shown that the combined constraint region of (4), viewed as a standard optimization problem with the corporate unit controlling all the variables, is composed of piecewise linear equality constraints.

Theorem 2

Problem (4) is equivalent to maximizing the linear function $C^0 x$ over a feasible region, call it $\bar{S}$, comprised of S^0 and piecewise linear equality constraints.

Proof

To begin, let us use Theorem 1 to rewrite (4):

$$\max_{x,w} C^0 x \qquad (5.1)$$

subject to

$$A^0 x^0 \leqq b^0 \qquad (5.2)$$

$$\left.\begin{aligned} w^i a^i &= c^i && (5.3)\\ w^i(a^i x^i + a^{\bar i} x^{\bar i} - b^i) &= 0 && (5.4)\\ a^i x^i + a^{\bar i} x^{\bar i} &\leqq b^i && (5.5)\\ w^i &\geqq 0 && (5.6) \end{aligned}\right\} \quad i = 1, \ldots, M$$

where $C^i \equiv (c^i, c^{\bar i})$ and $A^i \equiv [a^i, a^{\bar i}]$. From (5.3) it can be seen that w^i must be a vertex of the polyledron $W^i = \{\omega^i : \omega^i a^i = c^i\}$. This implies, along with (5.5) and (5.6) that (5.4) can be expressed as:

$$\max\ (w^i(a^i x^i + a^{\bar i} x^{\bar i} - b^i) : w^i \in W^i) = 0 \qquad (6)$$

or equivalently as:

$$c^i x^i + P(x^{\bar i}) = 0, \qquad i = 1, \ldots, M \qquad (7)$$

where

$$P(x^{\bar i}) \equiv \max\ (w^i(a^{\bar i} x^{\bar i} - b^i) : w^i \in W^i)$$

is a piecewise linear, concave continuous function of $x^{\bar i}$. Therefore, (5) can be written as:

$$\max\ (C^0 x : x \in S^0,\ c^i x^i + P(x^{\bar i}) = 0,\ i = 1, \ldots, M).$$

Because $C^0 x$ is a linear function and bounded above on S by, say, $\max(C^0 x : x \in S)$, the following can be concluded.

Corollary 2

The solution to (4) lies at a vertex of $\bar{S}$.

Next, we will show that the solution vertex of $\bar{S}$ is also a vertex of S. This result points up the reason why linear programming techniques can be effectively applied to solve problem (4).

Theorem 3

The solution $(x^*) \equiv (x^{*i}, x^{*\bar i})$ to the linear BMPP occurs at a vertex of S.

Proof

To begin, let $(x^{i1}, x^{\bar i 1}), \ldots, (x^{iq}, x^{\bar i q})$ be the distinct vertices of S. Since any point in S may be written as a convex combination of these vertices, let:

$$(x^{*i}, x^{*\bar i}) = \sum_{j=1} \lambda_j (x^{\bar i j}).$$

where

$$\sum_{j=1}^{\bar q} \lambda_j = 1;$$

$$\lambda_j > 0, \quad j = 1, \ldots, \bar q, \quad \bar q \leqq q.$$

It must be shown that $\bar q = 1$. To see this, let us write (7) at the solution $(x^{*i}, x^{*\bar{i}})$ for all $i = 1, \ldots, M$:

$$\begin{aligned} 0 &= P(x^{*\bar{i}}) + c^i x^{*i} \\ &= P\left(\sum_j \lambda_j x^{\bar{i}j}\right) + c^i \sum_j \lambda_j x^{ij} \\ &\geqq \sum_j \lambda_j P(x^{\bar{i}j}) - \sum_j \lambda_j x^{ij}, \end{aligned}$$

by concavity of $P(x^{\bar{i}})$,

$$= \sum_j \lambda_j (P(x^{\bar{i}j}) + c^i x^{ij}).$$

Now, because $(x^{ij}, x^{\bar{i}j})$ is in S for all i and j, it follows from either (6) or (7) that:

$$P(x^{\bar{i}j}) + c^i x^{ij} \geqq 0, \; i = 1, \ldots, M; \; j = 1, \ldots, \bar{q}. \quad (8)$$

Nothing that $\lambda_j > 0$, $j = 1, \ldots, \bar{q}$, the equality in (8) must hold lest a contradiction result in the above sequence. The implication from (7) is that $(x^{ij}, x^{\bar{i}j}) \in \bar{S}$, $j = 1, \ldots, \bar{q}$, and that $(x^{*i}, x^{*\bar{i}})$ can be written as a convex combination of points in $\bar{S}$. But because $(x^{*i}, x^{*\bar{i}})$ is a vertex of $\bar{S}$ it must be true that $\bar{q} = 1$.

The fact that each vertex of $\bar{S}$ is also a vertex of S leads to the following.

Corollary 3

$\bar{S}$ is formed from S^0 and the faces of S.

Lastly, a special case of (4) is presented as a corollary to Theorem 1.

Corollary 4

The linear BMPP can be written as a linear program when $A^i \equiv [a^i, \bar{a}^i]$ is given by the $m^i \times (n^i + n^{\bar{i}})$ matrix $[a^i, 0]$, $i = 1, \ldots, M$.

Proof

The result follows directly by first rewriting (4.2) in its equivalent form: $\max(c^i x^i : a^i x^i \leqq b^i - \bar{a}^i x^{\bar{i}})$, where c^i is the subvector of C^i corresponding to x^i, and then invoking duality theory. The complementarity term (3.4) may thus be written as $w^i(b^i - \bar{a}^i x^{\bar{i}}) - c^i x^i = 0$ which, by assumption, reduces to $w^i b^i - cx^i = 0$.